

Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions

Miao-Kun Wang^a, Yu-Ming Chu^a, Ye-Ping Jiang^b

^a Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China;

^b College of Mathematics and Econometrics, Hunan University, Changsha 410082, China.

Correspondence should be addressed to Yu-Ming Chu, chuyuming@hutc.zj.cn

Abstract: In this paper, a generalization of Ramanujan's cubic transformation, in the form of an inequality, is proved for zero-balanced Gaussian hypergeometric function $F(a, b; a + b; x)$, $a, b > 0$.

Keywords: Gaussian hypergeometric function, Ramanujan's cubic transformation, inequality

2010 Mathematics Subject Classification: 33C05

1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} \quad (1.1)$$

for $x \in (-1, 1)$, where (a, n) denotes the shifted factorial function $(a, n) = a(a+1)(a+2)\cdots(a+n-1)$ for $n = 1, 2, \dots$, and $(a, 0) = 1$ for $a \neq 0$. And $F(a, b; c; x)$ is called zero-balanced if $c = a + b$.

It is well known that $F(a, b; c; x)$ has many important applications in various fields of the mathematical and natural sciences [1-2], and many classes of special function in mathematical physics are particular cases of this function [3]. For a extensive list of $F(a, b; c; x)$ see [4-7].

As the special case of Gaussian hypergeometric function, for $r \in (0, 1)$, Legendre's complete elliptic integrals of the first kind is defined by

$$\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right).$$

Some of the most important properties of the elliptic integrals $\mathcal{K}(r)$ are the Landen identities:

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}(\sqrt{1-r^2}),$$

namely,

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4r}{(1+r)^2}\right) = (1+r)F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad (1.2)$$

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1-r}{1+r}\right)^2\right) = \frac{1+r}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-r^2\right). \quad (1.3)$$

For zero-balanced Gaussian hypergeometric functions $F(a, b; a+b; x)$, $a, b > 0$, Simić and Vuorinen [8] determined maximal region of ab plane where equations (1.2) and (1.3) turn on respective inequalities valid for each $x \in (0, 1)$.

As is known to all, Ramanujan's cubic transformation is defined as

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left(\frac{1-r}{1+2r}\right)^3\right) = (1+2r)F\left(\frac{1}{3}, \frac{2}{3}; 1; r^3\right), \quad (1.4)$$

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-r}{1+2r}\right)^3\right) = \frac{1+2r}{3}F\left(\frac{1}{3}, \frac{2}{3}; 1; 1-r^3\right). \quad (1.5)$$

Inspired by the ideas of Simić and Vuorinen [8], we find the maximal region of ab plane for $F(a, b; a+b; x)$, $a, b > 0$ where equations (1.4) and (1.5) turn on respective inequalities valid for each $x \in (0, 1)$.

The following asymptotic formulas for zero-balanced hypergeometric function (see [9, 10]) will be used in this paper.

$$F(a, b; a+b; r) \sim -\frac{1}{B(a, b)} \log(1-r) \quad (1.6)$$

and

$$B(a, b)F(a, b; a+b; r) + \log(1-r) = R(a, b) + O((1-r) \log(1-r)), \quad (1.7)$$

as r tends to 1, where

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re} w > 0 \quad (1.8)$$

is the classical beta function,

$$R(a, b) = -\Psi(a) - \Psi(b) - 2\gamma, \quad R(1/3, 2/3) = \log 27, \quad (1.9)$$

$$\Psi(z) = \frac{d}{dz}(\log \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \operatorname{Re} z > 0, \quad (1.10)$$

and γ is the Euler-Mascheroni constant.

Lemma 1.1 (See [8, Lemma 1.1]). Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $b_n > 0$ for all $n \in \{0, 1, 2, \dots\}$. Let $h(x) = f(x)/g(x)$, then

(1) If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$;

(2) If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0 < n \leq n_0$ and (strictly) decreasing (increasing) for $n > n_0$, then there exists $x_0 \in (0, r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0, x_0)$ and (strictly) decreasing (increasing) on (x_0, r) .

2. Main Results

For convenience, we first introduce the following regions in $\{(a, b) \in \mathbb{R}^2 | a > 0, b > 0\}$:

$$\begin{aligned} D_1 &= \{(a, b) | a, b > 0, ab \leq 2/9, ab - \frac{2}{9}(a+b) \leq 0\}, \\ D_2 &= \{(a, b) | a, b > 0, ab < 2/9, ab - \frac{2}{9}(a+b) > 0\}, \\ D_3 &= \{(a, b) | a, b > 0, ab \geq 2/9, ab - \frac{2}{9}(a+b) \geq 0\}, \\ D_4 &= \{(a, b) | a, b > 0, ab > 2/9, ab - \frac{2}{9}(a+b) < 0\}, \\ D_5 &= \{(a, b) | a, b > 0, a+b \leq 1, ab - \frac{2}{9}(a+b) \leq 0\}, \\ D_6 &= \{(a, b) | a, b > 0, a+b \geq 1, ab - \frac{2}{9}(a+b) \geq 0\}. \end{aligned}$$

Clearly, $D_1 \cup D_2 \cup D_3 \cup D_4 = \{(a, b) \in \mathbb{R}^2 | a > 0, b > 0\}$, $D_5 \subset D_1$ and $D_6 \subset D_3$.

Theorem 2.1. If $(a, b) \in D_1$, then the inequality

$$F(a, b; a+b; \frac{9r(1+r+r^2)}{(1+2r)^3}) \leq (1+2r)F(a, b; a+b; r^3) \quad (2.1)$$

holds for all $r \in (0, 1)$. Also, if $(a, b) \in D_3$, then the reversed inequality

$$F(a, b; a+b; \frac{9r(1+r+r^2)}{(1+2r)^3}) \geq (1+2r)F(a, b; a+b; r^3) \quad (2.2)$$

takes place for each $r \in (0, 1)$, with equality in each instance if and only if $(a, b) = (1/3, 2/3)$ or $(a, b) = (2/3, 1/3)$.

In the remaining region $(a, b) \in D_2 \cup D_4$, neither of the above inequalities holds for each $r \in (0, 1)$.

Theorem 2.2. If $(a, b) \in D_1$, then the double inequality

$$1 \leq \frac{(1+2r)F(a, b; a+b; r^3)}{F(a, b; a+b; \frac{9r(1+r+r^2)}{(1+2r)^3})} \leq \frac{\sqrt{3}B(a, b)}{2\pi} \quad (2.3)$$

holds for all $r \in (0, 1)$. And if $(a, b) \in D_3$, then inequality (2.3) is reversed

$$\frac{\sqrt{3}B(a, b)}{2\pi} \leq \frac{(1+2r)F(a, b; a+b; r^3)}{F(a, b; a+b; \frac{9r(1+r+r^2)}{(1+2r)^3})} \leq 1. \quad (2.4)$$

Moreover, both bounds in inequalities (2.3) and (2.4) are sharp and each equality is reached for $a = 1/3$ and $b = 2/3$, or $a = 2/3$ and $b = 1/3$.

Corollary 2.3. For $r \in (0, 1)$, and $(a, b) \in D_1$, one has

$$\frac{2\pi}{\sqrt{3}} \frac{1}{B(a, b)} F(a, b; a + b; r^3) < F(a, b; a + b; \frac{9r(1+r+r^2)}{(1+2r)^3}) < 3F(a, b; a + b; r^3). \quad (2.5)$$

In the region $(a, b) \in D_3$, one has

$$F(a, b; a + b; r^3) < F(a, b; a + b; \frac{9r(1+r+r^2)}{(1+2r)^3}) < \frac{6\pi}{\sqrt{3}} \frac{1}{B(a, b)} F(a, b; a + b; r^3). \quad (2.6)$$

Theorem 2.4. Let $B = B(a, b)$ and $R = R(a, b)$ are defined as in (1.8) and (1.9), respectively. Then for $(a, b) \in D_5$, inequality

$$0 \leq (1+2r)F(a, b; a+b; r^3) - F(a, b; a+b; \frac{9r(1+r+r^2)}{(1+2r)^3}) \leq \frac{2(R - \log 27)}{B} \quad (2.7)$$

holds for all $r \in (0, 1)$. Also, for $(a, b) \in D_6$,

$$0 \leq F(a, b; a + b; \frac{9r(1+r+r^2)}{(1+2r)^3}) - (1+2r)F(a, b; a + b; r^3) \leq \frac{2(R - \log 27)}{B}. \quad (2.8)$$

Theorem 2.5. For $(a, b) \in D_1$ and each $x \in (0, 1)$, one has

$$\frac{1}{3} \leq \frac{F(a, b; a + b; \left(\frac{1-x}{1+2x}\right)^3)}{(1+2x)F(a, b; a + b; 1-x^3)} \leq \frac{\sqrt{3}B(a, b)}{6\pi}. \quad (2.9)$$

(2) For $(a, b) \in D_3$ and each $x \in (0, 1)$, one has

$$\frac{\sqrt{3}B(a, b)}{6\pi} \leq \frac{F(a, b; a + b; \left(\frac{1-x}{1+2x}\right)^3)}{(1+2x)F(a, b; a + b; 1-x^3)} \leq \frac{1}{3}. \quad (2.10)$$

(3) For $(a, b) \in D_5$ and each $x \in (0, 1)$, we have

$$\begin{aligned} (1+2x)F(a, b; a + b; 1-x^3) &\leq 3F(a, b; a + b; \left(\frac{1-x}{1+2x}\right)^3) \\ &\leq (1+2x) \left[F(a, b; a + b; 1-x^3) + \frac{2(R(a, b) - \log 27)}{B(a, b)} \right]. \end{aligned} \quad (2.11)$$

(4) For $(a, b) \in D_6$ and each $x \in (0, 1)$, we have

$$\begin{aligned} 0 &\leq (1+2x)F(a, b; a + b; 1-x^3) - 3F(a, b; a + b; \left(\frac{1-x}{1+2x}\right)^3) \\ &\leq \frac{2(1+2x)(\log 27 - R(a, b))}{B(a, b)}. \end{aligned} \quad (2.12)$$

3. Proofs of Theorems

In order to prove our main results, we introduce several symbols. Throughout this section, we let

$$F(x) = F(a, b; a + b; x), \quad G(x) = F(a, b; a + b + 1; x),$$

where $a, b > 0$ with $(a, b) \neq (1/3, 2/3)$ and $(a, b) \neq (2/3, 1/3)$, and

$$F^*(x) = F\left(\frac{1}{3}, \frac{2}{3}; 1; x\right), \quad G^*(x) = F\left(\frac{1}{3}, \frac{2}{3}; 2; x\right).$$

Lemma 3.1. (1) The function $f(r) = F(r)/F^*(r)$ is strictly decreasing in $(0, 1)$ on D_1 , and strictly increasing in $(0, 1)$ on D_3 . Moreover, if $(a, b) \in D_2$ (D_4 , resp.), then there exists r_0 (r_0^* , resp.) such that $f(r)$ is strictly increasing (decreasing, resp.) in $(0, r_0)$ ($(0, r_0^*)$, resp.), and strictly decreasing (increasing, resp.) in $(r_0, 1)$ ($(r_0^*, 1)$, resp.);

(2) The function $g(r) = G(r)/G^*(r)$ is strictly decreasing in $(0, 1)$ on D_5 and strictly increasing in $(0, 1)$ on D_6 .

Proof. For part (1), denote by $A_n = (a, n)(b, n)/[(a + b, n)n!]$ and $A_n^* = (1/3, n)(2/3, n)/[(n)!]^2$, then

$$f(r) = \frac{F(r)}{F^*(r)} = \frac{\sum_{n=0}^{\infty} A_n r^n}{\sum_{n=0}^{\infty} A_n^* r^n}. \quad (3.1)$$

Note that the monotonicity of $\{A_n/A_n^*\}$ depends on the sign of

$$H_n = (ab - \frac{2}{9})n + ab - \frac{2}{9}(a + b). \quad (3.2)$$

We divide the proof into four cases.

Case 1 $(a, b) \in D_1$. Then (3.2) implies $H_n < 0$ for $n = 0, 1, 2, \dots$, and $f(r)$ is strictly decreasing in $(0, 1)$ by (3.1) and Lemma 1.1.

Case 2 $(a, b) \in D_3$. Then (3.2) implies $H_n > 0$ for $n = 0, 1, 2, \dots$, and $f(r)$ is strictly increasing in $(0, 1)$ by (3.1) and Lemma 1.1.

Case 3 $(a, b) \in D_2$. Then from (3.2) we conclude that the sequence $\{A_n/A_n^*\}$ increases and then decreases. By (3.1) and Lemma 1.1(3), there exists $r_0 \in (0, 1)$ such that $f(r)$ is strictly increasing in $(0, r_0)$ and strictly decreasing in $(r_0, 1)$.

Case 4 $(a, b) \in D_4$. Then from (3.2) we know that the sequence $\{A_n/A_n^*\}$ decreases and then increases. By (3.1) and Lemma 1.1(3), there exists $r_0^* \in (0, 1)$ such that $f(r)$ is strictly decreasing in $(0, r_0^*)$ and strictly increasing in $(r_0^*, 1)$.

For part (2), denote by $B_n = (a, n)(b, n)/[(a + b + 1, n)n!]$ and $B_n^* = (1/3, n)(2/3, n)/[(2, n)(n)!]$, then

$$g(r) = \frac{G(r)}{G^*(r)} = \frac{\sum_{n=0}^{\infty} B_n r^n}{\sum_{n=0}^{\infty} B_n^* r^n}. \quad (3.3)$$

Note that the monotonicity of $\{B_n/B_n^*\}$ depends on the sign of

$$H_n^* = (a + b + ab - \frac{11}{9})n + \frac{2}{9}(9ab - a - b - 1). \quad (3.4)$$

We divide the proof into two cases.

Case A $(a, b) \in D_5$. Then $a + b + ab - 11/9 \leq 11(a + b)/9 - 11/9 \leq 0$ and $9ab - a - b - 1 = 9ab - 2(a + b) + (a + b) - 1 \leq 0$. Thus $H_n^* < 0$ for $n = 0, 1, 2, \dots$ (because $(a, b) \neq (1/3, 2/3)$ and $(a, b) \neq (2/3, 1/3)$) by (3.4). Therefore, $g(r)$ is strictly decreasing in $(0, 1)$ follows from (3.3) and Lemma 1.1.

Case B $(a, b) \in D_6$. Then $a + b + ab - 11/9 \geq 11(a + b)/9 - 11/9 \geq 0$ and $9ab - a - b - 1 = 9ab - 2(a + b) + (a + b) - 1 \geq 0$. Thus $H_n^* > 0$ for $n = 0, 1, 2, \dots$ by (3.4). Therefore, $g(r)$ is strictly increasing in $(0, 1)$ follows from (3.3) and Lemma 1.1.

Proof of Theorem 2.1. Let $x = x(r) = r^3$ and $y = y(r) = 9r(1 + r + r^2)/(1 + 2r)^3$, then simple computation leads to $0 < x < y < 1$ for $0 < r < 1$. Using Lemma 3.1(1), we get $f(x) > f(y)$ on D_1 , and $f(x) < f(y)$ on D_3 .

For $(a, b) \in D_1$, by (1.4), one has

$$\frac{F(r^3)}{F^*(r^3)} > \frac{F(y)}{F^*(y)}, \quad F(y) < \frac{F^*(y)}{F^*(r^3)} F(r^3) = (1 + 2r)F(r^3).$$

Thus inequality (2.1) follows.

Inequality (2.2) is obtained analogously. The remaining conclusions easily follows from Lemma 3.1(1).

Proof of Theorem 2.2. Let $f(r)$ be defined as in Lemma 3.1(1), then $f(r)$ is strictly decreasing on D_1 . Then (1.6) leads to

$$1 = \lim_{r \rightarrow 0^+} \frac{F(r)}{F^*(r)} > \frac{F(r)}{F^*(r)} > \lim_{r \rightarrow 1^-} \frac{F(r)}{F^*(r)} = \frac{B(1/3, 2/3)}{B(a, b)} = \frac{2\sqrt{3}\pi}{3B(a, b)}$$

and

$$\frac{\sqrt{3}B(a, b)}{2\pi} \frac{1}{F^*(y(r))} > \frac{1}{F(y(r))} \implies \frac{\sqrt{3}B(a, b)}{2\pi} \frac{F^*(x(r))}{F^*(y(r))} > \frac{F(x(r))}{F(y(r))}.$$

Thus inequality (2.3) is clear.

Inequality (2.4) valid on D_3 can be proved similarly.

Lemma 3.2. The function

$$J(r) = (1 + 2r^{1/3})F(a, b; a + b; r) - F(a, b; a + b; \frac{9r^{1/3}(1 + r^{1/3} + r^{2/3})}{(1 + 2r^{1/3})^3})$$

is strictly increasing in $(0, 1)$ on D_5 and strictly decreasing in $(0, 1)$ on D_6 .

Proof. Let $z = 9r^{1/3}(1 + r^{1/3} + r^{2/3})/(1 + 2r^{1/3})^3$. Then

$$1 - z = \frac{(1 - r^{1/3})^3}{(1 + 2r^{1/3})^3}, \quad \frac{dz}{dr} = \frac{3(1 - r^{1/3})^2}{r^{2/3}(1 + 2r^{1/3})^4}.$$

Note that

$$(1-x)F(a+1, b+1; a+b+1; x) = F(a, b; a+b+1; x).$$

Differentiating $J(r)$ gives

$$\begin{aligned} r^{2/3}(1-r^{1/3})J'(r) &= \frac{2}{3}(1-r^{1/3})F(a, b; a+b; r) + \frac{ab}{a+b} \frac{r^{2/3}(1+2r^{1/3})(1-r^{2/3})}{1-r} \\ &\quad \times F(a, b; a+b+1; r) - \frac{3ab}{(a+b)(1+2r^{1/3})} F(a, b; a+b+1; z) \\ &= \frac{2}{3}(1-r^{1/3})F(r) + \frac{ab}{a+b} \frac{r^{2/3}(1+2r^{1/3})(1-r^{2/3})}{1-r} G(r) \\ &\quad - \frac{3ab}{(a+b)(1+2r^{1/3})} G(z). \end{aligned} \quad (3.5)$$

On the other hand, differentiating Ramanujan cubic transformation, we get

$$\frac{2}{3} \frac{G^*(z)}{1+2r^{1/3}} = \frac{2}{3}(1-r^{1/3})F^*(r) + \frac{2}{9} \frac{r^{2/3}(1+2r^{1/3})(1-r^{2/3})}{1-r} G^*(r). \quad (3.6)$$

Let $g(r)$ be defined as in Lemma 3.1(2), then $g(r)$ is strictly decreasing in $(0, 1)$ on D_5 . Then from $0 < r < z < 1$ we get $g(r) > g(z)$, namely

$$G(z) < \frac{G^*(z)}{G^*(r)} G(r). \quad (3.7)$$

Equations (3.5) and (3.6) together with inequality (3.7) yield

$$\begin{aligned} r^{2/3}(1-r^{1/3})J'(r) &> \frac{2}{3}(1-r^{1/3})F(r) + \frac{ab}{a+b} \frac{r^{2/3}(1+2r^{1/3})(1-r^{2/3})}{1-r} G(r) \\ &\quad - \frac{3ab}{(a+b)(1+2r^{1/3})} \frac{G^*(z)}{G^*(r)} G(r) \\ &= \frac{2}{3}(1-r^{1/3})F(r) + \frac{ab}{a+b} \frac{r^{2/3}(1+2r^{1/3})(1-r^{2/3})}{1-r} G(r) - \frac{3ab}{(a+b)(1+2r^{1/3})} \\ &\quad \times \left((1-r^{1/3})(1+2r^{1/3}) \frac{F^*(r)}{G^*(r)} + \frac{1}{3} \frac{r^{2/3}(1+2r^{1/3})^2(1-r^{2/3})}{1-r} \right) G(r) \\ &= \frac{2}{3}(1-r^{1/3})F(r) - \frac{3ab}{(a+b)} (1-r^{1/3}) \frac{F^*(r)}{G^*(r)} G(r) \\ &= \frac{2}{3}(1-r^{1/3}) \left[F(r) - \frac{9ab}{2(a+b)} \frac{F^*(r)}{G^*(r)} G(r) \right]. \end{aligned}$$

Note that

$$\frac{F'(r)}{F^{*'}(r)} = \frac{9ab}{2(a+b)} \frac{G(r)}{G^*(r)}.$$

Thus

$$\frac{3}{2} r^{2/3} J'(r) > F(r) - \frac{F'(r)}{F^{*'}(r)} F^*(r) = \frac{F^2(r)}{F^{*'}(r)} \left(\frac{F^*(r)}{F(r)} \right)'.$$

It follows from Lemma 3.1(1) and the fact that $D_5 \subset D_1$ that $(F^*(r)/F(r))' \geq 0$ on D_5 . Hence $J'(r) > 0$, and $J(r)$ is strictly increasing in $(0, 1)$ on D_5 .

Since $g(r)$ is strictly decreasing in $(0, 1)$ on D_6 , we have $g(r) > g(z)$, namely

$$G(z) > \frac{G^*(z)}{G^*(r)} G(r).$$

With the similar argument, one has

$$\frac{3}{2} r^{2/3} J'(r) < \frac{F^2(r)}{F^{*'}(r)} \left(\frac{F^*(r)}{F(r)} \right)' < 0,$$

since $f(r) = F(r)/F^*(r)$ is strictly increasing in $(0, 1)$ on $D_6 \subset D_3$. Hence $J(r)$ is strictly decreasing in $(0, 1)$ on D_6 . \square

Proof of Theorem 2.4. By Lemma 3.2 we obtain $\lim_{r \rightarrow 0^+} J(r) < J(r) < \lim_{r \rightarrow 1^-} J(r)$ on D_5 and $\lim_{r \rightarrow 1^-} J(r) < J(r) < \lim_{r \rightarrow 0^+} J(r)$ on D_6 . Clearly, $\lim_{r \rightarrow 0^+} J(r) = 0$. And by (1.7), we have

$$\begin{aligned} & \lim_{r \rightarrow 1^-} J(r) \\ &= \lim_{r \rightarrow 1^-} \frac{3R(a, b) - 3 \log(1 - r) - (R(a, b) - 3 \log[(1 - r^{1/3})/(1 + 2r^{1/3})]) + o(1)}{B(a, b)} \\ &= \frac{2(R(a, b) - \log 27)}{B(a, b)}. \end{aligned}$$

The assertion of Theorem 2.4 follows.

Proof of Theorem 2.5. Theorem 2.5 follows from Theorems 2.2 and 2.4 with $x = (1 - r)/(1 + 2r) \in (0, 1)$.

Acknowledgement. This research was supported by the Natural Science Foundation of China (Grant Nos. 11071059, 11071069, 11171307), and the Innovation Team Foundation of the Department of Education of Zhejiang Province (Grant no. T200924).

References

- [1] R. Askey, Handbooks of special functions, A Century of Mathematics in America, Part III (P. Duren. ed.), Amer. Math. Soc., Providence. RI, 1989, pp. 369-391.
- [2] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge Univ. Press, Cambridge, 2010.
- [3] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series, Vol. 3: More Special Functions, Gordon & Breach, New York, 1988.

- [4] M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.
- [5] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Sons, New York, 1997.
- [6] S.-L. Qiu and M. Vuorinen, Special functions in geometric function theory, in: Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier Sci. B. V., Amsterdam, 2005, pp. 621-659.
- [7] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy, and M. Vuorinen, Inequalities for zero-balanced hypergeometric functions, Trans. Amer. Math. Soc., 1995, 347, pp. 1713-1723.
- [8] S. Simić, M. Vuorinen, Landen inequalities for zero-balanced hypergeometric functions, Abstr. Appl. Anal., 2012, Art. ID 932061, 11 pages.
- [9] B. C. Berndt, Ramanujan's Notebooks, Part I, Springer-Verlag, New York, 1985.
- [10] R. J. Evans, Ramanujan's second notebooks: asymptotic expansions for hypergeometric series and related functions. Ramanujan revisited (Urbana-Champaign, Ill., 1987), 537-560, Academic Press, Boston, MA, 1988.